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A New Method to Prove Several Inequalities

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Abstract: As is known to all, to strengthen and refine the famous arithmetic-geometric-harmonic mean inequality H (w, a) ≤ G (w, a) ≤ A (w, a), it has been becoming the focus of the theoretical research of inequality from estimating the mean difference. the difference between the two inequalities mentioned above is estimated by using variance, and these results are strengthened or generalized by using a consistent proof model.

Keywords: Arithmetic Mean; Geometric Mean; Compressed Independent Variables Theorem; Inequality

1.Introduction

In this paper, we assume that $n\in N, n\geq 2$, $a=(a_1,a_2,\cdots,a_n)\in R^n$, $w=(w_1,w_2,\cdots,w_n)\in[0,1]^n$, and $\sum_{i=1}^n w_i=1$ $A(w,a)=\sum_{i=1}^n w_i a_i$ is the weighted arithmetic mean of a, where $w_i\geq 0$ $i=1,2,\cdots,n$ is the weight coefficient. For $a\in(0,+\infty)^n$, the power mean of a with weight p is

$$M_p(w,a)=\begin{cases}\left(\sum_{i=1}^n w_i a_i\right)^{1/p}, & p\neq 0\\ \prod_{i=1}^n a_i^{w_i}, & p=0\end{cases}$$

In particular, $A(w,a)=M_1(w,a)$, $G(w,a)=M_0(w,a)$ and $H(w,a)=M_{-1}(w,a)$ are the weighted arithmetic, geometric, harmonic means of a respectively.

When $w_1=w_2=\cdots=w_n=1/n$, we denote $M_p(w,a)$, $A(w,a)$, $G(w,a)$ and $H(w,a)$ by $M_p(a)$, $A(a)$, $G(a)$ and $H(a)$ respectively. Also denote $\bar{M}_p(a)=(M_p(a),M_p(a),\cdots,M_p(a))$,and similarly denote $\bar{A}(a)$, $\bar{G}(a)$ and $\bar{H}(a)$.

As is known to all, to strengthen and refine the famous arithmetic-geometric- harmonic mean inequality

$$H(w,a)\leq G(w,a)\leq A(w,a) \tag{1}$$

has been becoming the focus of the theoretical research of inequality from estimating the mean difference.

Let $0<m<M$, $a=(a_1,a_2,\cdots,a_n)\in[m,M]^n$, then there are some beautiful conclusions about the upper and lower bounds with the mean difference of $A(a)-G(a)$. Such as in paper "Analytic Inequalities" [1],

$$\frac{1}{n-1}\min_{1\leq i\leq n}\{w_i\}\sum_{1\leq i<j\leq n}\left(a_i^{1/2}-a_j^{1/2}\right)^2\leq A(w,a)-G(w,a)$$
$$\leq\max_{1\leq i\leq n}\{w_i\}\sum_{1\leq i<j\leq n}\left(a_i^{1/2}-a_j^{1/2}\right)^2 \tag{2}$$

and

$$\frac{1}{1-\min_{1\leq i\leq n}\{w_i\}}\sum_{1\leq i<j\leq n}w_iw_j\left(a_i^{1/2}-a_j^{1/2}\right)^2\leq A(w,a)-G(w,a)$$
$$\leq\frac{1}{\min_{1\leq i\leq n}\{w_i\}}\sum_{1\leq i<j\leq n}w_iw_j\left(a_i^{1/2}-a_j^{1/2}\right)^2 \tag{3}$$

In "A refinement of the arithmetic mean-geometric mean inequality" and "Handbook of Means and Their Inequalities" [2, 3], hold

$$\frac{1}{2M}\sum_{i=1}^n w_i\left(a_i-A(w,a)\right)^2\leq A(w,a)-G(w,a)\leq\frac{1}{2m}\sum_{i=1}^n w_i\left(a_i-A(w,a)\right)^2 \tag{4}$$

and

$$\frac{1}{4M}\left(\sum_{i=1}^n w_ia_i^2-G^2(w,a)\right)\leq A(w,a)-G(w,a)\leq\frac{1}{4m}\left(\sum_{i=1}^n w_ia_i^2-G^2(w,a)\right) \tag{5}$$

The results in "Problem 247" and "Epecaric J and Fink A classical and new inequalities in analysis" [4, 5] be equivalent to

$$\frac{\sum_{1\leq i<j\leq n}\left(a_i-a_j\right)^2}{2n^2M}\leq A(a)-G(a)\leq\frac{\sum_{1\leq i<j\leq n}\left(a_i-a_j\right)^2}{2n^2m} \tag{6}$$

In fact, it is the special case of (4) when $w_1=w_2=\cdots=w_n=\frac{1}{n}$. In "A New refinement of the arithemetic mean-geometic mean inequality" [6], the author also obtain

$$\frac{1}{2M}\sum_{i=1}^n w_i\left(a_i-G(w,a)\right)^2\leq A(w,a)-G(w,a)\leq\frac{1}{2m}\sum_{i=1}^n w_i\left(a_i-G(w,a)\right)^2 \tag{7}$$

Mercer in "Improved upper and lower bounds for the difference of An-Gn" [7] respectively strengthen (4) and (7) as follows

$$\frac{M-G(w,a)}{2M\left(M-A(w,a)\right)}\sum_{i=1}^n w_i\left(a_i-A(w,a)\right)^2\leq A(w,a)-G(w,a)$$
$$\leq\frac{G(w,a)-m}{2m\left(A(w,a)-m\right)}\sum_{i=1}^n w_i\left(a_i-A(w,a)\right)^2 \tag{8}$$

and

$$\frac{M-G(w,a)}{2M\left(M-G(w,a)\right)-2G(w,a)\left(A(w,a)-G(w,a)\right)}\cdot\sum_{i=1}^{n}w_i\left(a_i-G(w,a)\right)^2$$

$$\le\frac{G(w,a)-m}{2m\left(G(w,a)-m\right)+2G(w,a)\left(A(w,a)-G(w,a)\right)}\sum_{i=1}^{n}w_i\left(a_i-G(w,a)\right)^2 \quad (9)$$

There are many other forms of inequality on $A-G$, readers can refer to "Analytic Inequalities" [1] and "Improved upper and lower bounds for the difference of An-Gn" [7] to "Some Refinements of Ky Fan's Inequality" [8].

In "Sierpinski's inequality" [9], Alzer, H prove

$$\frac{n-1}{n}A(a)+\frac{1}{n}H(a)\ge G(a) \quad (10)$$

In "A new method to prove and find analytic inequalities" [10], above result was intensified as follows: let $r=\frac{n^2}{n^2+4n-4}$ then

$$rA(a)+(1-r)H(a)\ge G(a) \quad (11)$$

In this paper, we will strengthen or popularize (4)-(7) and (10), (11) by using aunanimous model of proof, partial results have the similar intensity with (8) and (9), but whose form is more concise.

2. Relevant Result

Let $D\subseteq R^n$ is a symmetric convex set containing inner point, $i=1,2,\cdots,n$, denote

$$\breve{D}_i=\left\{x\in D\mid x_i=\max_{1\le k\le n}\{x_k\}\right\}-\{x\in D\mid x_1=x_2=\cdots=x_n\}$$

and

$$\hat{D}_i=\left\{x\in D\mid x_i=\min_{1\le k\le n}\{x_k\}\right\}-\{x\in D\mid x_1=x_2=\cdots=x_n\}.$$

Lemma 2.1 Let $I\subseteq(0,+\infty)$ function $f:I^n\overset{Def.}{=}D\to R$ is continuous symmetric, and has continuous partial derivative. If $\partial f/\partial x_1>(<)\partial f/\partial x_2$ permanent holds in $\breve{D}_1\cap\hat{D}_2$, then for $\forall a\in I^n, f(a)\ge(\le)f(\overline{A}(a))$, and the equality holds if and only if $a_1=a_2=\cdots=a_n$.

If take the functional transformation in lemma2. 1, we obtain the lemma 2.2 and lemma 2.3 as follows, and the proof can refer to "Problem 395" [11].

Lemma 2.2 Let $I\subseteq(0,+\infty)$, function $f:I^n\overset{Def.}{=}D\to R$ is symmetric, and has continuous partial derivative.

If $x_1\partial f/\partial x_1>(<)x_2\partial f/\partial x_2$ permanent holds in $\breve{D}_1\cap\hat{D}_2$, then for $\forall a\in I^n, f(a)\ge(\le)f(\overline{G}(a))$ and the equality holds if and only if $a_1=a_2=\cdots=a_n$.

Proof. Set

$\ln I^n=\left\{\ln a=(\ln a_1,\ln a_2,\cdots\ln a_n)\mid a\in I^n\right\}$

$g:y\in\ln I^n\to f\left(e^{y_1},e^{y_1},\cdots,e^{y_n}\right)$,

Then $\frac{\partial g}{\partial y_1}=\frac{\partial f}{\partial x_1}\cdot\frac{dx_1}{dy}=e^{y_1}\frac{\partial f}{\partial x_1}=x_1\frac{\partial f}{\partial x_1}$, $\frac{\partial g}{\partial y_2}=\frac{\partial f}{\partial x_2}\cdot\frac{dx_2}{dy}=e^{y_2}\frac{\partial f}{\partial x_2}=x_2\frac{\partial f}{\partial x_2}$,

When $y=\ln x\in\ln\left[\breve{D}_1\cap\hat{D}_2\right]$, hold $\frac{\partial g}{\partial y_1}-\frac{\partial g}{\partial y_2}=x_1\frac{\partial f}{\partial x_1}-x_2\frac{\partial f}{\partial x_2}>(<)0$.

According to lemma 2.1, for $\ln a\in\ln I^n$, $g(\ln a)\ge(\le)g\left(\overline{A}(a)\right)$ that is $f(a)\ge(\le)f\left(\overline{G}(a)\right)$.

Lemma 2.3 Let $p\ne0$, $I\subseteq(0,+\infty)$, function $f:I^n\overset{Def.}{=}D\to R$ is symmetric, and has continuous partial derivative. If $x_1^{1-p}\frac{\partial f}{\partial x_1}-x_2^{1-p}\frac{\partial f}{\partial x_2}>(<)0$.

Permanent holds in $\breve{D}_1\cap\hat{D}_2$, then for $\forall a\in I^n$, $f(a)\ge(\le)f(\overline{M}_P(a))$, and the equality holds if and only if $a_1=a_2=\cdots=a_n$.

Proof.

Let $\left(I^n\right)^p=\left\{x^p\mid x\in I^n\right\}$,

$g:y\in\left(I^n\right)^p=f\left(y_1^{1/p},y_2^{1/p},\cdots,y_n^{1/p}\right)$.

$\frac{g(y)}{\partial y_1}=\frac{1}{p}y_1^{\frac{1-p}{p}}\frac{\partial f}{\partial x_1}=\frac{1}{p}x_1^{1-p}\frac{\partial f}{\partial x_1}$, $\frac{g(y)}{\partial y_2}=\frac{1}{p}y_2^{\frac{1-p}{p}}\frac{\partial f}{\partial x_2}=\frac{1}{p}x_2^{1-p}\frac{\partial f}{\partial x_2}$.

when $p>0$, we get that g satisfy $\frac{\partial g}{\partial y_1}-\frac{\partial g}{\partial y_2}>(<)0$ on $\left(\breve{D}_i\right)^p$ and $\left(\hat{D}_j\right)^p$.

when $p<0$, thanks to $\breve{D}_i^p=\widehat{\left(D^p\right)}_i$, so $\frac{\partial g}{\partial y_2}-\frac{\partial g}{\partial y_1}=-\frac{1}{p}\left(x_1^{1-p}\frac{\partial f}{\partial x_1}-x_2^{1-p}\frac{\partial f}{\partial x_2}\right)>0$

According to lemma2.1, for $\forall$ $a\in I^n$, hold $a^p\in\left(I^n\right)^p$ and $g\left(a^p\right)\ge g\left(\overline{A}\left(a^p\right)\right)$, that is $f(a)\ge(\le)f(\overline{M}_P(a))$.

3. Several Upper and Lower Bounds of $A(a)-G(a)$

Theorem 3.1

$$\frac{\sum_{1\le i<j\le n}(a_i-a_j)^2}{2n[(n-1)M+G(a)]}\le A(a)-G(a)\le\frac{\sum_{1\le i<j\le n}(a_i-a_j)^2}{2n[(n-1)m+G(a)]}\quad(12)$$

that is

$$\frac{\sum_{i=1}^n(a_i-A(a))^2}{2[(n-1)M+G(a)]}\le A(a)-G(a)\le\frac{\sum_{i=1}^n(a_i-A(a))^2}{2[(n-1)m+G(a)]}\quad(13)$$

Proof. It is easy to prove $\sum_{i=1}^n(a_i-A(a))^2=\frac{1}{n}\sum_{1\le i<j\le n}(a_i-a_j)^2$,so (12) is equivalent to (13).

Let

$$f:a\in[m,M]^n\to\frac{\sum_{1\le i<j\le n}(a_i-a_j)^2}{2n[(n-1)m+G(a)]}-A(a)+G(a),$$

then

$$\frac{\partial f}{\partial a_1}=\frac{\sum_{i=2}^n(a_1-a_i)}{n[(n-1)m+G(a)]}-\frac{G(a)\sum_{1\le i<j\le n}(a_i-a_j)^2}{2n^2a_1[(n-1)m+G(a)]^2}-\frac{1}{n}+\frac{1}{na_1}G(a)$$

$$a_1\frac{\partial f}{\partial a_1}=\frac{a_1\sum_{i=2}^n(a_1-a_i)}{n[(n-1)m+G(a)]}-\frac{G(a)\sum_{1\le i<j\le n}(a_i-a_j)^2}{2n^2[(n-1)m+G(a)]^2}-\frac{a_1}{n}+\frac{1}{n}G(a),$$

$$=\frac{(n-1)a_1^2-a_1\sum_{i=2}^n a_i}{n[(n-1)m+G(a)]}-\frac{G(a)\sum_{1\le i<j\le n}(a_i-a_j)^2}{2n^2[(n-1)m+G(a)]^2}-\frac{a_1}{n}+\frac{1}{n}G(a),$$

and

$$a_1\frac{\partial f}{\partial a_1}-a_2\frac{\partial f}{\partial a_2}=\frac{(a_1-a_2)}{n[(n-1)m+G(a)]}\left[(n-1)(a_1+a_2)-\sum_{i=3}^n a_i-(n-1)m-G(a)\right].$$

When $a\in\breve{D}_1\cap\hat{D}_2\cap[m,M]^n$,obtain

$$a_1\frac{\partial f}{\partial a_1}-a_2\frac{\partial f}{\partial a_2}\ge\frac{(a_1-a_2)}{n[(n-1)m+G(a)]}\left[(n-1)(a_1+a_2)-(n-2)a_1-(n-1)m-a_1^{\frac{n-1}{n}}a_2^{\frac{1}{n}}\right]$$

$$=\frac{(a_1-a_2)}{n[(n-1)m+G(a)]}\left[a_1^{\frac{n-1}{n}}\left(a_1^{\frac{1}{n}}-a_2^{\frac{1}{n}}\right)+(n-1)(a_2-m)\right]>0.$$

According to lemma 2.2, for $\forall\ a\in[m,M]^n$, hold $f(a)\ge f(\bar{G}(a))$,and

$$\frac{\sum_{1\le i\le j\le n}^{n}(a_i-a_j)^2}{2n[(n-1)m+G(a)]}-A(a)+G(a)$$

$$\ge\frac{\sum_{1\le i\le j\le n}^{n}(A(a)-G(a))^2}{2n[(n-1)m+G(\bar{G}(a))]}-A(\bar{G}(a))+G(\bar{G}(a)$$

$$=0\ .$$

this is the right of (12). Similarly, we can prove the left of (13).

Remark 3.2 (12) is stronger than (6) apparently, when $w_1=w_2=\cdots w_n=\frac{1}{n}$ (13) is stronger than (4) apparently, and is as strong as (8). Next, we will prove (4) using (13). If (i=1,2...n) all are rational numbers, we may as well assume that $w_i(i=1,2,\cdots,n)$ have the same denominator, denote $w_i=\frac{t_i}{T}$, then $\sum_{i=1}^n w_i=1$ and $T=\sum_{i=1}^n t_i$,For vector $\left(\underbrace{a_1,\cdots,a_1}_{t_1},\underbrace{a_2,\cdots,a_2}_{t_2},\cdots\underbrace{a_n,\cdots,a_n}_{t_n}\right)$,

by (13), we get that

$$\frac{\sum_{i=1}^n t_i\left(a_i-\frac{1}{T}\sum_{i=1}^n t_ia_i\right)^2}{2\left[(T-1)M+\sqrt[T]{\prod_{i=1}^n a_i^{t_i}}\right]}\le\frac{1}{T}\sum_{i=1}^n t_ia_i-\sqrt[T]{\prod_{i=1}^n a_i^{t_i}}\le\frac{\sum_{i=1}^n t_i\left(a_i-\frac{1}{T}\sum_{i=1}^n t_ia_i\right)^2}{2\left[(T-1)m+\sqrt[T]{\prod_{i=1}^n a_i^{t_i}}\right]},$$

$$\frac{\sum_{i=1}^n t_i\left(a_i-\sum_{i=1}^n w_ia_i\right)^2}{2TM}\le\sum_{i=1}^n w_ia_i-\prod_{i=1}^n a_i^{w_i}\le\frac{\sum_{i=1}^n t_i\left(a_i-\sum_{i=1}^n w_ia_i\right)^2}{2Tm},$$

$$\frac{1}{2M}\sum_{i=1}^n w_i\left(a_i-\sum_{i=1}^n w_ia_i\right)^2\le\sum_{i=1}^n w_ia_i-\prod_{i=1}^n a_i^{w_i}\le\frac{1}{2m}\sum_{i=1}^n w_i\left(a_i-\sum_{i=1}^n w_ia_i\right)^2.$$

That is (4). For $\forall w_i$ (i = 1, 2, ..., n), as irrational number is the limit of rational number, so (4) hold as usual.

Theorem 3.3

$$\frac{\sum_{1\le i<j\le n}(a_i-a_j)^2}{2n^2M^{(n-2)/(n-1)}A^{1/(n-1)}(a)}\le A(a)-G(a)\le\frac{\sum_{1\le i<j\le n}(a_i-a_j)^2}{2n^2m^{(n-1)/n}A^{1/n}(a)}\quad(14)$$

thai is

$$\frac{\sum_{i=1}^n(a_i-A(a))^2}{2nM^{(n-2)/(n-1)}\cdot A^{1/(n-1)}(a)}\le A(a)-G(a)\le\frac{\sum_{i=1}^n(a_i-A(a))^2}{2nm^{(n-1)/n}\cdot A^{1/n}(a)}\quad(15)$$

Proof. As $\sum_{i=1}^n(a_i-A(a))^2=\frac{1}{n}\sum_{1\le i<j\le n}(a_i-a_j)^2$, so(14) is equivalent to (15),we prove (14) as follows.

Let

$$f:a\in[m,M]^n\to 2n^2M^{\frac{n-2}{n-1}}A^{\frac{1}{n-1}}(a)(A(a)-G(a))-\sum_{1\le i\le j\le n}(a_i-a_j)^2,$$

then

$$\frac{\partial f}{\partial a_1}=\frac{2nM^{(n-2)/(n-1)}}{n-1}(A(a))^{1/(n-1)-1}(A(a)-G(a))$$

$$+2n^2M^{(n-2)/(n-1)}(A(a))^{1/(n-1)}\left(\frac{1}{n}-\frac{1}{na_1}G(a)\right)-2\sum_{2\le i\le n}(a_1-a_i),$$

and

$$\frac{\partial f}{\partial a_1}-\frac{\partial f}{\partial a_2}=2nM^{(n-2)/(n-1)}\cdot\frac{a_1-a_2}{a_1a_2}(A(a))^{1/(n-1)}G(a)-2n(a_1-a_2).$$

when $a\in\breve{D}_1\cap\hat{D}_2\cap[m,M]^n$, obtain

$$\frac{\partial f}{\partial a_1}-\frac{\partial f}{\partial a_2}$$

$$=\frac{2n(a_1-a_2)}{a_1a_2}\left[M^{(n-2)/(n-1)}\left(\frac{1}{n}\sum_{i=1}^n a_i\right)^{1/(n-1)}\sqrt[n]{\prod_{i=1}^n a_i}-a_1a_2\right]$$

$$\ge\frac{2n(a_1-a_2)}{a_1a_2}\left[a_1^{(n-2)/(n-1)}\left(\frac{1}{n}a_1+\frac{n-1}{n}a_2\right)^{1/(n-1)}a_1^{1/n}a_2^{(n-1)/n}-a_1a_2\right]$$

$$>\frac{2n(a_1-a_2)}{a_1a_2}\left[a_1^{(n-2)/(n-1)}\cdot\left(a_1^{1/n}a_2^{(n-1)/n}\right)^{1/(n-1)}\cdot a_1^{1/n}a_2^{(n-1)/n}-a_1a_2\right].$$

$$=0$$

By lemma 2.1, for $\forall\ a\in[m,M]^n$, f (a) ≥ f (A(a)), that is the left of (14). Similarly, we can prove the right of (14).

Remark 3.4 (15) is stronger than (6), when $w_1 = w_2 = \cdots w_n = \frac{1}{n}$, (13) is stronger than (4) apparently, and is as strong as (8). Be similar to the proof of remark 3.2, we can prove (4) using (15).

Theorem 3.5 Let $k = \frac{2n}{n-1}$, then

$$A(a) - G(a) \le \frac{\sum_{i=1}^{n} \left(a_i - G(a)\right)^2}{(2n-k)m + kG(a)}. \tag{16}$$

Proof. Let

$$f : a \in [m, M]^n \to \frac{\sum_{i=1}^{n} \left(a_i - G(a)\right)^2}{(2n-k)m + kG(a)} - A(a) + G(a),$$

then

$$\frac{\partial f}{\partial a_1} = \frac{2\left(a_1 - G(a)\right)\left(1 - \frac{1}{na_1} G(a)\right) - \frac{2}{na_1} G(a) \sum_{i=2}^{n} \left(a_i - G(a)\right)}{(2n-k)m + kG(a)}$$

$$- \frac{k}{na_1} G(a) \frac{\sum_{i=1}^{n} \left(a_i - G(a)\right)^2}{\left[(2n-k)m + kG(a)\right]^2} - \frac{1}{n} + \frac{1}{na_1} G(a)$$

$$= \frac{2a_1 - 2G(a) - \frac{2}{na_1} G(a) \sum_{i=1}^{n} \left(a_i - G(a)\right)}{(2n-k)m + kG(a)}$$

$$- \frac{k}{na_1} G(a) \frac{\sum_{i=1}^{n} \left(a_i - G(a)\right)^2}{\left[(2n-2-k)m + kG(a)\right]^2} - \frac{1}{n} + \frac{1}{na_1} G(a),$$

and

$$a_1 \frac{\partial f}{\partial a_1} - a_2 \frac{\partial f}{\partial a_2} = \frac{2\left(a_1^2 - a_2^2\right) - 2\left(a_1 - a_2\right) G(a)}{(2n-k)m + kG(a)} - \frac{a_1 - a_2}{n}$$

$$= \frac{2\left(a_1 - a_2\right)}{n\left[(2n-k)m + kG(a)\right]} \left[n\left(a_1 + a_2\right) - \frac{n^2}{n-1} G(a) - \frac{n^2 - 2n}{n-1} m \right]$$

When $a \in \breve{D}_1 \cap \widehat{D}_2 \bigcap [m, M]^n$, we obtian

$$a_1 \frac{\partial f}{\partial a_1} - a_2 \frac{\partial f}{\partial a_2} \ge \frac{2\left(a_1 - a_2\right)}{n\left[(2n-k)m + kG(a)\right]} \left[n\left(a_1 + a_2\right) - \frac{n^2}{n-1} a_1^{\frac{n-1}{n}} a_2^{\frac{1}{n}} - \frac{n^2 - 2n}{n-1} a_2 \right]$$

$$= \frac{n^2}{n-1} \cdot \frac{2\left(a_1 - a_2\right)}{n\left[(2n-k)m + kG(a)\right]} \left(\frac{n-1}{n} a_1 + \frac{1}{n} a_2 - a_1^{\frac{n-1}{n}} a_2^{\frac{1}{n}} \right)$$

> 0.

Thanks to lemma 2.2, for $\forall\ a \in [m, M]^n$, $f(a) \ge f\left(\bar{G}(a)\right)$, that is the right of (16).

Remark 3.6 When $w_1 = w_2 = \cdots w_n = \frac{1}{n}$,(16) is stronger than (7) apparently. Be similar to the proof of remark 3.1, we can prove (7) using (16).

Theorem 3.7

$$\frac{\frac{1}{n} \sum_{i=1}^{n} a_i^2 - G^2(a)}{2M + 2G(a)} \le A(a) - G(a) \le \frac{\frac{1}{n} \sum_{i=1}^{n} a_i^2 - G^2(a)}{2m + 2G(a)}. \tag{17}$$

Proof. Let

$$f : a \in [m, M]^n \to \frac{\frac{1}{n} \sum_{i=1}^{n} a_i^2 - G^2(a)}{2m + 2G(a)} - A(a) + G(a),$$

then

$$\frac{\partial f}{\partial a_1} = \frac{a_1 - \frac{1}{a_1} G^2(a)}{n\left(m + G(a)\right)} - \frac{2G(a)}{na_1} \cdot \frac{\frac{1}{n} \sum_{i=1}^{n} a_i^2 - G^2(a)}{\left(2m + 2G(a)\right)^2} - \frac{1}{n} + \frac{G(a)}{na_1},$$

and

$$a_1 \frac{\partial f}{\partial a_1} - a_2 \frac{\partial f}{\partial a_1} = \frac{a_1 - a_2}{n\left(m + G(a)\right)} \left[a_1 + a_2 - m - G(a) \right].$$

When $a \in \breve{D}_1 \cap \widehat{D}_2 \bigcap [m, M]^n$, we obtian

$$a_1 \frac{\partial f}{\partial a_1} - a_2 \frac{\partial f}{\partial a_1} \ge \frac{a_1 - a_2}{n\left(m + G(a)\right)} \left(a_1 + a_2 - m - a_1^{\frac{n-1}{n}} a_2^{\frac{1}{n}} \right)$$

$$= \frac{a_1 - a_2}{n\left(m + G(a)\right)} \left[a_1^{\frac{n-1}{n}} \left(a_1^{\frac{1}{n}} - a_2^{\frac{1}{n}} \right) + \left(a_2 - m\right) \right] > 0.$$

— By lemma 2.2, for $\forall\ a \in [m, M]^n$, $f(a) \ge f\left(\bar{G}(a)\right)$, that is the left of (17). Similarly, we can prove the right of (17).

Using the proof of remark 3.2, by the theorem 3.7, we get the following corollary:

Corollary 3.8

$$\frac{\sum_{i=1}^{n} w_i a_i^2 - G^2(w,a)}{2M + 2G(w,a)} \le A(w,a) - G(w,a) \le \frac{\sum_{i=1}^{n} w_i a_i^2 - G^2(w,a)}{2m + 2G(w,a)} \tag{18}$$

Remark 3.9 (18) is stronger than (5).

4. A Stronger Inequality (Alzer, H.)

In "Sierpinski's inequality" [9], Alzer, H proved: $\frac{n-1}{n} A(a) + \frac{1}{n} H(a) \ge G(a)$. In "A new method to prove and find analytic inequalities" [12], above result was intensified as follows: let, $r = n^2 / \left(n^2 + 4n - 4\right)$,then $rA(a) + (1-r) H(a) \ge G(a)$. In this section, we will strengthen the inequality to the following theorem 4.1.

Theorem 4.1 Let $r = \frac{(n-1)^{\frac{n-1}{n}}}{n}$, then

$$rA(a) + (1-r) H(a) \ge G(a). \tag{19}$$

Proof. When n = 2, it is easy to prove the conclusion. When $n \ge 3$, let

$$f : a \in (0, +\infty)^n \to qA(a) + (1-q) H(a) - G(a),$$

where q > r. Then

$$\frac{\partial f}{\partial a_1} = \frac{q}{n} + (1-q) \frac{n}{a_1^2 \left(\sum_{i=1}^{n} a_i^{-1}\right)^2} - \frac{1}{na_1} G(a),$$

and

$$a_1^2 \frac{\partial f}{\partial a_1} - a_2^2 \frac{\partial f}{\partial a_2} = \left(a_1 - a_2\right) \left[\frac{q}{n} \left(a_1 + a_2\right) - \frac{1}{n} G(a) \right]$$

When $a \in \breve{D}_1 \cap \widehat{D}_2$,we obtian

$$a_1^2\frac{\partial f}{\partial a_1}-a_2^2\frac{\partial f}{\partial a_2}>(a_1-a_2)\left[\frac{r}{n}(a_1+a_2)-\frac{1}{n}a_1^{\frac{n-1}{n}}a_2^{\frac{1}{n}}\right]$$
$$=\frac{1}{n^2}a_2(a_1-a_2)\left[(n-1)^{\frac{n-1}{n}}(t+1)-nt^{\frac{n-1}{n}}\right]$$
$$\overset{Def.}{=}=\frac{1}{n^2}a_2(a_1-a_2)g(t)$$

where $t=\frac{a_1}{a_2}>1$, $g(t)=(n-1)^{\frac{n-1}{n}}(t+1)-nt^{\frac{1}{n}}$.

As

$$g'(t)=(n-1)^{\frac{n-1}{n}}-(n-1)t^{\frac{-1}{n}}=t^{\frac{-1}{n}}(n-1)^{\frac{n-1}{n}}\left[t^{\frac{1}{n}}-(n-1)^{\frac{1}{n}}\right]$$

then $g(t)$ is decreasing in $(1,n-1)$, is increasing in $(n-1,+\infty)$, and $g(t)$ attain the minimum value when $t=n-1$. That is

$$g(n-1)=(n-1)^{\frac{n-1}{n}}(n-1+1)-n(n-1)^{\frac{n-1}{n}}=0.$$

so when $a\in\breve{D}_1\cap\hat{D}_2$, $a_1^2\frac{\partial f}{\partial a_1}-a_2^2\frac{\partial f}{\partial a_2}>0$ hold.

By lemma 2.3($p=-1$), for $\forall\ a$, we get that

$$f(a)\ge f\left(\bar{M}_{-1}(a)\right)=f\left(\bar{H}(a)\right),$$

that is

$$qA(a)+(1-q)H(a)-G(a)\ge 0.$$

Let $q\to r$, the theorem 4.1 hold.

Remark 4.2 When $2\le n\le 27$, it is easy to prove $\frac{(n-1)^{\frac{n-1}{n}}}{n}\le\frac{n^2}{n^2+4n-4}$ by computer. When $n\ge 28$, as

$e^4\le 2n$, $e^{4-\frac{4}{n}}\le n\left(1+\frac{1}{n-1}\right)^{n-1}$,

$\left[\left(1+\frac{4n-4}{n^2}\right)^{\frac{n^2}{4n-4}}\right]^{4-\frac{4}{n}}\le n\left(1+\frac{1}{n-1}\right)^{n-1}$, $\left(\frac{n^2+4n-4}{n^2}\right)^n\le n\left(\frac{n}{n-1}\right)^{n-1}$,

$$\frac{(n-1)^{\frac{n-1}{n}}}{n}\le\frac{n^2}{n^2+4n-4}.$$

so (19) is stronger than (10) and (11).

5. Conclusion

In this paper, we use this method to strengthen and refine the famous arithmetic-geometric-harmonic mean inequality, by employing variance to estimate the difference between above two inequality, and to strengthen or popularize these results by using a unanimous model of proof.

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