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An Analysis of Teaching the Determinant Concept Using a 'Problem Chain' Approach Based on Problem-Based Learning

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Abstract: The teaching of linear algebra is challenging since its abstractness. This paper analyses the design of a "problem chain" teaching model based on problem-based learning for teaching the determinant concept in linear algebra. Through a step-by-step sequence of problems, students are guided from solving linear equation systems to understanding the definition, properties, and applications of determinants, enabling a deeper comprehension of the core concept in linear algebra from both algebraic and geometric perspectives. According to the theoretical foundation analyse, the problem chain teaching model focuses on student-centered learning, emphasizing active knowledge construction and inquiry. Using the determinant concept as an example, this paper demonstrates the design method and implementation of the problem chain model, providing theoretical and practical references for teaching reform in linear algebra and other mathematics courses.

Keywords: Problem-Based Learning; Linear Algebra; Concept of Determinant; Problem Chain; Teaching Design

1. Introduction

"Linear Algebra" is a fundamental course for science and engineering majors, as well as some economics and management programs in higher education institutions. With the rapid development of computer applications, big data and artificial intelligence technologies, the importance of the Linear Algebra course has grown more significantly. Many frontier technical problems are applications of linear algebraic theories and methods[1]. Among these, the determinant is a core concept in Linear Algebra. It serves as a critical tool for analyzing matrix properties and determining the case of solution(s) for linear equation systems. Due to its abstractness, it is often one of the most challenging topics for students to understand, making it a focal and difficult point in linear algebra teaching[2].

The concept of determinants involves fundamental but abstract notions such as permutations and inversion numbers[3]. Building its complex mathematical definition from these notions can often leave students confused, unable to grasp its mathematical significance or application value. Furthermore, the determinant is associated with many properties. These properties are challenging for students to memorize and apply flexibly, often leading to confusion. When computing higher-order determinants, students frequently struggle with selecting appropriate and efficient methods among various options, leaving them at a loss during calculations.

Traditional teaching methods often focus on the theoretical knowledge and the procedure of calculation, but they may lack the necessity of concept generation and how it generated when introducing the concept of determinants[4]. This approach can make it difficult for students to overcome the learning bottlenecks associated with mastering determinant knowledge and fails to effectively address their conceptual understanding barriers. As a result, students may end up memorizing the definition mechanically without understanding its underlying rationale or origins[5]. For instance, when teaching the properties of determinants, the traditional teaching approach often involves merely listing these properties without guiding students to explore the principles behind them or the logical connections between them. Similarly, in teaching computation methods, there is often a lack of comparative analysis through diverse examples, which would help students understand the appropriate contexts and advantages of different methods.

Therefore, in the teaching of determinant concept, it becomes particularly important to skillfully design problem-based scenarios to introduce concepts, helping students deeply understand the concepts and the necessity of the determinant. The "problem chain" teaching model for mathematics concepts, driven by problem-based learning, focuses on proposing and analyzing effective problems to guide students in thinking and exploration, thereby facilitating the effective formation and comprehension of new concepts. In this paper, the "problem chain" teaching model for mathematics concepts and its theoretical foundation are analyzed. It proposes a design method for implementing the "problem chain" teaching model and illustrates a detailed design by taking the teaching of the determinant concept as an example.

2. "Problem Chain" Teaching Model Based on Problem-Based Learning for Mathematical Concepts and Its Theoretical Foundation

Problem-based learning of mathematical concepts emphasizes starting from problems and guiding students to gradually approach the essence of the concept, enabling them to deeply understand it through the continuous process of solving problems. Problem-based learning centers on students and uses carefully designed, logically coherent, and progressively structured problem chains to encourage active exploration, critical thinking, and knowledge construction. In teaching mathematical concepts, this approach is particularly well-suited for understanding abstract concepts. It effectively stimulates students' interest in learning, helps them build an intuitive understanding, and gradually transitions them to rigorous logical expression. As early as 1980, the renowned American mathematician Halmos published a famous article titled "The Heart of Mathematics" in the "American Mathematical Monthly"[6], pointing out that axioms, theorems, proofs, concepts, definitions, theories, formulas, and methods are all merely essential components of mathematics, rather than its heart; it is problems that constitute the heart of mathematics. This shows the important of problem solving in teaching mathematics. The core of problem-based learning lies in using problems to stimulate thinking and using thinking to promote knowledge construction. In teaching mathematical concepts, a problem chain can break down complex knowledge into several closely connected smaller problems, guiding students to grasp the essence and scope of the concept from concrete to abstract and from specific to general. This approach not only focuses on knowledge transmission but also emphasizes cultivating students' mathematical thinking and problem-solving abilities[7].

The "problem chain" teaching model for mathematical concepts is reflected in several influential mathematical teaching theories. Constructivism suggests that learning is a process in which students actively construct knowledge[8]. In teaching mathematical concepts, the design of problem chains should align with students' existing knowledge structures. By creating well-designed problems that provoke cognitive conflict, students are guided toward actively constructing new knowledge. Vygotsky's theory of the zone of proximal development (ZPD)[9] posits that learning should go beyond students' current cognitive level while remaining within an acceptable range of challenge. The design of problem chains should therefore take into account students' cognitive abilities, progressively increasing the difficulty of problems to help students improve their skills step by step under the teacher's guidance. In Polya's problem-solving theory[10], the "four-step process" (understanding the problem, devising a plan, executing the plan, and reviewing the solution) aligns closely with the problem chain teaching method. In teaching mathematical concepts, teachers can design problem chains to guide students through this process, helping them gradually master the concept.

3. Design of Teaching Determinant Concept Using "Problem Chain" Teaching Model Based on Problem-Based Learning

The concept of determinants is a core topic in linear algebra. Due to its complexity and abstractness, students often feel confused during learning. In this section, using the "problem chain" teaching model, the instruction on the concept of determinants can be designed as a clear and structured exploratory process. This approach helps students gradually understand the essence, properties, and applications of determinants through problem-solving. In this section, the problem chain and the design intention of the problems for teaching the concept of determinant will proposed.

Problem 1.1 Please use elimination method to solve the following two-variables linear equations.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases} \quad (1)$$

Design Intention This question helps students to learning new content from solving linear equations in two variables, which is a topic familiar from high school. By introducing this problem, the teacher guides students from solving equation systems with specific numerical coefficients to solving systems with abstract coefficients. This progression ultimately leads to the general form of the unique solution for a system of equations, laying the foundation for the introduction of determinants.

Problem 1.2 Does the unique solution

$$\begin{cases} x_1 = \dfrac{b_1 a_{22} - a_{12} b_2}{a_{11}a_{22} - a_{12}a_{21}} \\ x_2 = \dfrac{a_{11} b_2 - b_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}} \end{cases} \quad (2)$$

always exist? When will it not exist?

Design Intention Deriving the general solution in Problem 1.1, it allow students to feel a sense of accomplishment. Then by presenting Problem 1.2, the teacher encourages reflection and critical thinking about the results obtained, creating a progressive effect that further stimulates students' interest in learning and fosters rigor of mathematics. Through guided thinking and teacher’s support, students can discover that when the denominator equals zero, a unique solution does not exist.

Problem 1.3 The value of the denominator determines whether the system of equations has a unique solution. Therefore, it is crucial to consider whether the denominator equals zero. It is necessary to assign a name to the expression for the denominator. Then what would be an appropriate name?

Design Intention Through the design of this problem, the teacher can guide students to experience from the specific expression for solutions to an abstract understanding of the concept of determinants. In English, the term "determinant" as the name of determinant, highlights the role of the determinant in determining whether a system of equations has a unique solution, emphasizing its important application. In Chinese, it uses the term "row-column Equation" to name determinant, reflecting the algebraic nature of determinants as expressions involving rows and columns. This name is both visual and indicative of the determinant's mathematical essence as an algebraic expression, enhancing students' deeper conceptual understanding. During teaching, teachers can discuss and compare these two naming approaches with students. This process not only sparks students' interest in learning but also helps them gain a more profound understanding of the concept.

Through the first set of problems above, the concept of the second-order determinant has been constructed. Following this, the concept will be extended to higher-order determinants.

Problem 2.1 We obtained the concept of second-order determinant through a system of linear equations with two variables. Can it be extended to third-order determinant? Is there a similar relationship between third-order determinants and systems of linear equations with three variables?

Design Intention Students have already understood the concept and algebraic role of second-order determinants through the process of solving systems of linear equations with two variables. Building on this foundation, they are guided through analogy to explore, whether the concept and properties of second-order determinants can be extended to third-order determinants and to further investigate the relationship between third-order determinants and systems of linear equations with three variables. Through this process, students not only grasp the method for extending determinants but also recognize the unifying nature and significance of determinants across different dimensions of linear algebra problems. During the process of derivation and analysis, students develop logical reasoning skills and learn how to generalize mathematical concepts from the specific to the general and from lower dimensions to higher dimensions, fostering curiosity and a spirit of exploration. Additionally, third-order determinants serve as an important bridge for defining higher-order determinants. By guiding students through this question, they not only understand the connection between second-order and third-order determinants but also gain an initial understanding of the recursive concept, laying the groundwork for subsequent learning of higher-order determinants and their properties.

Problem 2.2 What are the patterns in the calculation of third-order determinants? How can the equation for a third-order determinant be remembered?

Design Intention The expression for a third-order determinant is relatively complex and cannot be remembered through rote memorization alone. This question helps students draw analogies to second-order determinants, summarize the algebraic characteristics of third-order determinants, and derive the "diagonal rule" for calculation. This lays the foundation for understanding the definition of higher-order determinants.

Through the first and second sets of questions, the concepts of second-order and third-order determinants were constructed, leading to the introduction of the core concept, which is determinants.

Problem 3.1 We have gained the concepts of second-order and third-order determinants. Can these be extended to fourth-order, fifth-order, and n th-order determinants?

Design Intention Through this problem, students realize that the extension of determinants from lower-order to higher-order is a recursive process. It guides them to actively derive and summarize the recursive definition of higher-order determinants based on existing knowledge. This approach allows students to gain a deeper understanding of new concepts rather than passively accepting complex formulas. However, directly solving this problem may be challenging for students. Therefore, to address this issue, the teacher can introduce Problem 3.2 to help students think further.

Problem 3.2 What common patterns exist between second-order and third-order determinants? Why are the signs on the main diagonal positive, while those on the secondary diagonal are negative?

Design Intention Students are already familiar with the calculation rules for second-order and third-order determinants, but they may only be focused on the specific calculations and have not deeply understood the common patterns behind them. Through this problem, the teacher guides students to further explore the origin of the determinant's signs and the nature of the permutation rules, helping them gain a deeper understanding of the mathematical logic and unity of determinants. By comparing the calculation formulas for second-order and third-order determinants, students are led to discover the structural similarities between them: (1) each term is a product of n elements, with each element located in a different row and column; (2) when the row indices are arranged in order and the column indices take all possible for permutations, then all terms are generated, totaling $n!$ terms; (3) when the row indices are in order, the sign of each term depends on the number of inversions in the column index permutation.

By guiding students to compare the common patterns of second-order and third-order determinants and exploring the origin of the sign rules, this problem helps them gain an in-depth understanding of the logical structure of determinants from the perspective of permutations and parity. It also fosters their mathematical abstraction and reasoning abilities, encourages them to learn the mathematical concept of generalizing from the specific to the general, strengthens their ability of abstract and generalize, and sparks their interest in exploring mathematics.

Problem 3.3 Can the general definition of an n -order determinant be derived by analogy?

Design Intention This problem echoes Problem 3.1. Through the previous exploration, students can naturally derive the general definition of an n th-order determinant. At this point, the n th-order determinant that students construct is the result of their own induction and reasoning, developed through the process of generalizing from the specific to the general. This helps them gain a deeper understanding of the structure of determinants and increases their acceptance of the concept. At the same time, it cultivates students' abstract thinking and the ability to make analogies and transfer knowledge.

Problem 3.4 Does the fourth-order determinant have a diagonal rule?

Design Intention Through this problem, students are helped to strengthen their ability to distinguish and understand concepts.

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Determinants of order four and higher cannot be calculated using the diagonal rule, which is also a common mistake point for students when they first learn to compute determinants.

Problem 3.5 How do you understand the definition of an n th-order determinant? Is it possible to understand the definition of a determinant from both algebraic and geometric perspectives?

Design Intention Students often focus on the computational methods and rules when learning determinants, but they may not deeply understand the algebraic and geometric significance behind them, tending to view determinants as merely computational tools. In fact, from an algebraic perspective, the determinant is an important tool for analyzing and studying systems of linear equations, matrices, and linear transformations. The computation of determinants can also be understood as a function of a square matrix, and specifically as a linear functional with linear properties, which is a key component of the determinant's properties.

From a geometric perspective, a second-order determinant represents the directed area of a parallelogram, and a third-order determinant represents the directed volume of a parallelepiped. Through this geometric intuition, students can understand that a determinant is not just an algebraic computational tool, but also an important measure of "size" and "direction" in multidimensional space. At the same time, the algebraic and geometric meanings of determinants are interconnected. The determinant reflects how a matrix, as a linear transformation, changes the "scale" (expansion or contraction factor) and "direction" of a spatial region. The sign (positive or negative) represents whether the geometric transformation preserves or reverses the direction, while the value corresponds to the rate of volume change. As students continue their study of matrices, they will gain a deeper understanding of the definition of determinants.

Through the above three sets of problem chains, students can progressively deepen their understanding of the definition and computation methods of determinants, as well as their algebraic and geometric characteristics. Next, through the computation of determinants, the properties of determinants and the expansion rules can be naturally introduced.

4. The Logical Design of the Problem Chain for Teaching the Concept of Determinants

The design of a problem chain for teaching the concept of determinants follows a step-by-step and student-centered approach, ensuring logical progression from familiar concepts to abstract definitions. It aims to foster deep understanding through exploration and discovery, rather than passive reception of knowledge. Below is the logical structure in the design.

Phase 1: Introducing the concept through background context. It start with a familiar mathematical context, solving a system of linear equations with two variables. This provides an intuitive entry point for students to recognize the necessity of determinants in determining the existence and uniqueness of solutions.

Phase 2: Constructing the concepts of 2x2 and 3x3 determinants. In this phase, students progress from specific to general. The problem chain begin with 2x2 determinants derived from solving 2-variable linear systems; then gradually extend to 3x3 determinants, encouraging students to explore similarities and differences; finally highlight patterns, such as diagonal multiplication and the role of signs.

Phase 3: Generalizing to Higher-Order Determinants. Lead students to generalize the determinant's definition through analogy, transitioning from low-dimensional examples (2x2 and 3x3) to n-dimensional determinants. This is the phase in analyzing problems and constructing tools for solving problems.

Phase 4: Exploring the Properties and Significance of Determinants.

This four phases are follow the "Posing a Problem—Investigating the Problem (Establishing Mathematical Tools or Models)—Solving the Problem—Summarizing and Reflecting" serves as the main framework for structuring the teaching content[11]. This logical design ensures that the problem chain not only scaffolds learning effectively but also deepens students' conceptual and operational mastery of determinants. It promotes the development of higher-order thinking skills, such as abstraction, reasoning, and problem-solving.

5. Conclusions

This paper takes the teaching of the determinant concept as an example to explore the design methodology of the "problem chain" teaching model based on problem-based learning. By structuring problems progressively, the teaching process helps students from solving linear equations to understanding determinants' definitions, properties, and applications. The "problem chain" teaching model is student-centered, carefully designing problem scenarios to stimulate cognitive conflict, guiding students to actively explore, think, and construct knowledge. In solving problems, students not only learn the definition, properties, and calculation methods of determinants but also develop abstract thinking, analogy, and transfer skills, as well as mathematical reasoning from the specific to the general. This approach effectively overcomes the challenges students face in understanding abstract concepts in traditional teaching, making the learning process more logical and coherent.

The problem-based "problem chain" teaching model provides a practical approach for teaching complex concepts in linear algebra and serves as a reference for teaching reforms in other mathematics courses. In the future, further integration with information technology and teaching practices can optimize the design and implementation of "problem chains," enhancing the theories of mathematics education while fostering students' core mathematical literacy and innovative capabilities.

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