$L_2 - L_\infty$ Filtering for Constant Time-Delay Neural Networks with Quantized Output

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Abstract: This article mainly focuses on the $L_2 - L_\infty$ filtering problem of a class of constant delay neural networks with quantized outputs. The main purpose of this article is to design a quantized $L_2 - L_\infty$ filter that makes the filtering error system asymptotically stable in the absence of external disturbances, while achieving the required $L_2 - L_\infty$ interference suppression index with zero initial conditions. To this end, by constructing an appropriate Lyapunov–Krasovskii functional that fully captures the delay information and quantization effects, and by employing several advanced inequality scaling techniques, a set of sufficient conditions is derived to ensure the desired $L_2 - L_\infty$ performance of the filtering error system. Subsequently, using systematic decoupling methods, an explicit linear matrix inequality (LMI)-based design procedure is presented for the required $L_2 - L_\infty$ filter under constant time delays. The filter gain matrix can be obtained efficiently by solving the derived LMIs with standard numerical solvers. Numerical examples ultimately demonstrate that the proposed filter design method is not only effective, but also has lower conservatism. In summary, this work provides a systematic and computationally tractable approach for designing quantized $L_2 - L_\infty$ filters for constant time-delay neural networks.

Keywords: Neural Network Model; Steady Time Delay; Quantitative Output; $L_2 - L_\infty$ Performan

1. Introduction

Based on the unique connection mechanism, neural networks possess functional features such as strong robustness, distortion resistance, self-adaptation, and self-learning. In the 1940s, building upon the neuron doctrine established by Cajal, research on neural networks drew significant attention from fields such as computer science, information science, and neuroscience. Zhou [1] et al. designed a reduced-order filter for discrete-time delayed Markov jump neural networks to ensure stochastic stability and prescribed $L_2 - L_\infty$ performance, and provided delay-dependent sufficient conditions based on linear matrix inequalities (LMIs). Choi [2] et al. designed $L_2 - L_\infty$ filters for T-S fuzzy neural networks based on Wirtinger-type integral inequalities. Yang [3] et al. studied the $L_2 - L_\infty$ state estimation for continuous-time stochastic neural networks with time delays. They proposed a memory event-triggered strategy that utilizes historical data to optimize transmission and reduce communication frequency. As a branch of modern control theory, the $L_2 - L_\infty$ performance can adjust the expected output value to minimize it [4]. In addition, for further results on filtering problems for neural networks with time-varying delays, refer to the reference [5,6]. This control strategy can effectively handle systems with external disturbances and keep the energy peak gain within a given range. This motivation has driven the exploration of $L_2 - L_\infty$ filtering problems in time-delay neural networks with quantified outputs in this study.

This paper investigates the $L_2 - L_\infty$ filtering problem for time-delay neural networks with quantized output. The option is to design a quantized $L_2 - L_\infty$ filter for the constant time-delay case such that the error system achieves a predefined $L_2 - L_\infty$ performance under zero initial conditions. First, by establishing a reasonable Lyapunov–Krasovskii functional and selecting inequality scaling techniques, a criterion for the filtering error system to possess $L_2 - L_\infty$ performance is derived. Then, through several decoupling techniques, a design method for the $L_2 - L_\infty$ filter is proposed to guarantee that the filtering error system has $L_2 - L_\infty$ filtering performance.

Finally, the proposed $L_2 - L_\infty$ filter is validated through numerical simulations.

2. Preliminaries

The symbols involved in the subsequent sections are explained in detail in Table 1.

Table 1. Symbols

Symbols	Meaning
$\mathbb{R}^{n\times m}$	real matrix with n rows and columns
$\mathbb{R}^n$	n-dimensional Euclidean space
S_+^n	n- dimensional positive definite matrix
S^n	$n\times n$ real symmetric matrix
A^{-1}	inverse matrix of A
A^T	transpose matrix of A
$He(A)$	$A + A^T$
I	appropriate dimensional identity matrix
$\lVert x\rVert$	Euclidean norm of vector x
$L_2[0,\infty)$	the space of square integrable infinite-dimensional vector functions over interval $[0,\infty)$
$\sup(\cdot)$	supremum
j_i	n-dimensional column vector, the i-th row is 1 and the remaining rows are all 0
diag$\{a_1, a$	$[a_1, 0,0; 0, a_2, 0; 0,0, a_3]$

Lemma 1 [7] Given matrix $E \in S_+^n$, for a differentiable function $\eta \in \mathbb{R}^n$ on the interval $[j,k]$, the inequality

$$\int_j^k \dot{\eta}^T(u)X\dot{\eta}(u)du \geq \frac{1}{k-j}\Omega^T diag\{E, 3E, 5E\}\Omega \quad (1)$$

holds, where

$$\Omega = \begin{bmatrix} \eta(k) - \eta(j) \\ \eta(k) + \eta(j) - \frac{2}{k-j}\int_j^k \eta(a)da \\ \eta(k) - \frac{6}{k-j}\int_j^k \delta_{j,k}(a)\eta(a)da - \eta(j) \end{bmatrix}, \delta_{j,k}(a) = 2\left(\frac{a-j}{k-j}\right) - 1. \quad (2)$$

Lemma 2 [8] Take any real matrices X_1 and X_2 which have appropriate dimensionality, there exists a constant $y > 0$ makes the following holds

$$X_1X_2^T + X_2X_1^T \leq \frac{1}{y}X_1X_1^T + aX_2X_2^T. \quad (3)$$

Lemma 3 [9] Take matrices A_1, A_2, A_3, which have appropriate dimensions,

$$\begin{bmatrix} A_1 & A_2 \\ * & A_3 \end{bmatrix} < 0, \quad (4)$$

if and only if $A_1 < 0, A_3 - A_2^T A_1^{-1} A_2 < 0$, or $A_3 < 0, A_1 - A_2 A_3^{-1} A_2^T < 0$.

For neural networks with constant time delay, there are:

$$\dot{o}(s) = Wo(s) + Z_1\xi(o(s)) + Z_2\xi(o(s-h)) + T_1\varpi(s) \quad (5)$$

$$\beta(s) = Q_1 o(s) + Q_2 o(s-h) + T_2\varpi(s) \quad (6)$$

where $o(t) = [o_1(s)\cdots o_n(s)]^T \in \mathbb{R}^n, \beta(s) = [\beta_1(s)\cdots\beta_m(s)]^T \in \mathbb{R}^m$ 和 $\varpi(t) \in \mathbb{R}^m$ denote the neuron state, system output, and external disturbance input belonging to $L_2[0,\infty)$, respectively; h represents the constant time delay; $\mathrm{W} = \mathrm{diag}\{-\mathrm{w}_1, \cdots, -\mathrm{w}_n\} \in \mathbb{R}^{n\times n}$ is a self−feedback matrix with $\mathrm{w}_i > 0, i = 1, \dots, n$; $\mathrm{Z}_1 \in \mathbb{R}^{n\times n}$ and $\mathrm{Z}_2 \in \mathbb{R}^{n\times n}$ are the connection weight matrices; the known parameter matrices $\mathrm{T}_1 \in \mathbb{R}^{n\times m}, \mathrm{Q}_1 \in \mathbb{R}^{m\times n}, \mathrm{Q}_2 \in \mathbb{R}^{m\times n}, \mathrm{T}_2 \in \mathbb{R}^{m\times m}$; $\xi(o(s)) = [\xi_1(o(s))\cdots\xi_n(o(s))]^T \in \mathbb{R}^n \to \mathbb{R}^n$ represent the activation function that satisfies the Lipschitz condition: : there exists a Lipschitz constant $L_\xi > 0$ that satisfies the following condition:

$$\lVert\xi(b_2) - \xi(b)\rVert \leq L_\xi\lVert b_2 - b_1\rVert, \forall b_1, b_2 \in \mathbb{R}^n. \quad (7)$$

This study incorporates quantitative impact into the framework and assumes that the neural network output is transmitted to the filter through an unreliable communication network. The expression of the target filter is as follows:

$$\dot{\breve{o}}(s) = W\breve{o}(s) + Z_1\xi(\breve{o}(s)) + Z_2\xi(\breve{o}(s-h)) + L\Lambda[\beta(s) - \breve{\beta}(s)] \quad (8)$$

$$\breve{\beta}(s) = Q_1\breve{o}(s) + Q_2\breve{o}(s-h) \quad (9)$$

where $\breve{o}(s) = [\breve{o}_1(s)\cdots\breve{o}_n(s)]^T \in \mathbb{R}^n$ denote the filter state and $\breve{\beta}(s) = [\breve{\beta}_1(s)\cdots\breve{\beta}_\mathrm{m}(s)]^T \in \mathbb{R}^m$ filter output; $L \in \mathbb{R}^{n\times m}$ is the filter gain and can be calculated; $\Lambda(\varsigma) = [\Lambda_1(\varsigma_1)\cdots\Lambda(\varsigma_m)]^T$ is a logarithmic quantizer, and

$$\Lambda_i(\varsigma_i) = \begin{cases} \Delta^l\psi_0, & \frac{1}{1+\rho}\Delta^l\psi_0 < \varsigma_i \leq \frac{1}{1-\rho}\Delta^l\psi_0, l = 0, \pm 1, \pm 2, \\ 0, & \varsigma_i = 0, \\ -\Lambda_i(-\varsigma_i), & \varsigma_i < 0, \end{cases} \quad (10)$$

$\psi_0 > 0$ is a scaling scalar and $\rho = \frac{1-\Delta}{1+\Delta}(0 < \Delta < 1)$ denotes the quantization density, when $i \in \{1, \dots, m\}$.

Using the sector bound approach [10], we can express the $\Lambda(\varsigma)$ as

$$\Lambda(\varsigma) = (1 + \iota)\varsigma, \quad (11)$$

and $\iota \in [-\rho, \rho]$. From equations (8) and (9), the filtering error system can be established:

$$\dot{\delta}(s) = \overline{W}_1\delta(s) - \overline{A}_2\delta(s-h) + Z_1 g(\delta(s)) + Z_2 g(\delta(s-h)) + \overline{T}\varpi(s), \quad (12)$$

$$\widetilde{\beta}(s) = Q_1\delta(s) + Q_2\delta(s-h) + T_2\varpi(s) \quad (13)$$

where

$\delta(s) = o(s) - \breve{o}(s), \quad \widetilde{\beta}(s) = \beta(s) - \breve{\beta}(s),$

$$g(\delta(s)) = \xi(\delta(s) + \breve{o}(s)) - \xi(\breve{o}(s)),$$
$$g(\delta(s-h)) = \xi(\delta(s-h) + \breve{o}(s-h)) - \xi(\breve{o}(s-h)),$$
$$\overline{W}_1 = W - (1+\iota)LT_1, \overline{W}_2 = (1+\iota)LQ_2, \overline{T} = T_1 - (1+\iota)LT_2.$$

3. Main Results

For the neural network (5)-(6), we find a quantized $L_2 - L_\infty$ filter as in equations (8) and (9) to make sure that the filtering error system in equations (12) and (13) has $L_2 - L_\infty$ performance; that is,

Definition 1: For a given disturbance attenuation index $\gamma > 0$, on the one hand, when $\varpi(t) \equiv 0$, the error system (12)-(13) is asymptotically stable; and on the other hand, when $\varpi(t) \neq 0$, with the zero initial conditions, for any $\varpi(t) \in L_2[0,\infty)$ and as $K \to \infty$, satisfy the following inequality:

$$\| \delta(s) \|_\infty^2 \leq \gamma^2 \| \varpi(s) \|_2^2, \forall s \geq 0, \quad (14)$$

where

$$\| \delta(s) \|_\infty^2 = sup_{s\geq 0}\{\delta^T(s)\delta(s)\}, \| \varpi(s) \|_2^2 = \int_0^K \varpi^T(s)\varpi(s)\,ds,$$

Then the mapping from $\varpi(s)$ to $\delta(s)$ is considered to have an L_2 gain less than or equal to γ.

Remark 1: The essence of the $L_2 - L_\infty$ approach lies in effectively suppressing the amplitude variations of the control variables, thereby achieving the smallest possible absolute values in the time-domain response to disturbance inputs and multiple-output systems, which in turn ensures robust stability of the system.

The analysis criteria and comprehensive design methods of quantization $L_2 - L_\infty$ filters will be introduced in this part. For systems (12)-(13), we can propose the following criteria for performance analysis of $L_2 - L_\infty$:

Theorem 1 Given a scalar $\gamma > 0$, suppose that there exists a matrix $P = (P_{ij})_{3\times 3} \in S_+^{3n}$, $Y, S_1, R_1 \in S_+^n, U_1, U_2 \in \mathbb{R}^{n\times n}$, and diagonal matrices $A_1, A_2 \in S_+^n$, such that

$$\begin{bmatrix} Y & I \\ * & \gamma^2 I \end{bmatrix} > 0, \quad (15)$$

$$\Xi_1 < 0, \quad (16)$$

where

$$\Xi_1 = He(N_1^T P N_0 + N_4^T \widehat{U} N_5 + J_1^T Y J_8) + \check{S}_1 + j_8^T h^2 R_1 j_8 - N_2^T \widetilde{R}_1 N_2 + N_3^T \hat{S}_2 N_3 - j_7^T j_7,$$

$$\hat{S}_1 = diag\{2S_1, -S_1, 0_{6n\times 6n}\}, \check{S}_1 = diag\{S_1, -S_1, 0_{6n\times 6n}\},$$
$$\hat{S}_2 = diag\{L_\phi^2 A_1, L_\phi^2 A_2, -A_1, -A_2\},$$
$$\widetilde{R}_1 = diag\{R_1, 3R_1, 5R_1\}, \widehat{U} = [U_1 \quad U_2]^T,$$
$$N_0 = [j_8^T \quad j_1^T - j_2^T \quad j_1^T + j_2^T - 2j_3^T]^T, N_1 = [j_1^T \quad hj_3^T \quad hj_4^T]^T,$$
$$N_2 = [j_1^T - j_2^T \quad j_1^T + j_2^T - 2j_3^T \quad j_1^T - j_2^T - 6j_4^T]^T,$$
$$N_3 = [j_1^T \quad j_2^T \quad j_5^T \quad j_6^T]^T, N_4 = [j_8^T \quad j_1^T]^T,$$
$$N_5 = j_8 - \overline{W}_1 j_1 + \overline{W}_2 j_2 + Z_1 j_5 + Z_2 j_6 + \overline{T} j_7,$$
$$j_i = [0_{n\times(i-1)n} \quad I_n \quad 0_{n\times(8-i)n}]^T, \ i = 1, \dots, 8,$$

Then, under zero initial conditions, for all nonzero $\varpi(t) \in L_2[0,\infty)$ the error system (12)-(13) holds.

Proof: Define $\chi(s) = col\{\sigma_0(s), \sigma_1(s), \sigma_2(s)\}$, where (17)

$$\sigma_0(s) = [\delta^T(s) \quad \delta^T(s-h)]^T,$$
$$\sigma_1(s) = \frac{1}{h}\left[\int_{-h}^0 \delta_s^T(a)da \quad \int_{-h}^0 \iota(a)\delta_s^T(a)da\right]^T,$$
$$\sigma_2(s) = \left[g^T(\delta(s)) \quad g^T(\delta(s-h)) \quad w^T(s) \quad \dot{\delta}^T(s)\right]^T,$$
$$\delta_s(a) = \delta(a+s), \iota(a) = 2\frac{a+h}{h} - 1.$$

Next, using the free weighted matrix method, assign any matrix $U_1, U_2 \in \mathbb{R}^{n\times n}$, we have

$$\chi^T(s) He(N_4^T \widehat{U} N_5)\chi(s) = 2[\dot{\delta}^T(s)U_1 + \delta^T(s)U_2] \times [\dot{\delta}(s) - \overline{W}_1\delta(s) + \overline{W}_2\delta(s-h) - Z_1 g(\delta(s)) - Z_2 g(\delta(s-h)) - \overline{T}\varpi(s)] = 0. \quad (18)$$

Furthermore, by equation (7), for any diagonal matrix $A_1, A_2 \in S_+^n$, can get

$$L_\phi^2 \delta^T(s) A_1 \delta(s) - g^T(\delta(s)) A_1 g(\delta(s)) \geq 0,$$
$$L_\phi^2 \delta^T(s-h) A_2 \delta(s-h) - g^T(\delta(s-h)) A_2 g(\delta(s-h)) \geq 0 \quad (19)$$

and it can be transformed into

$$\chi^T(s) N_3^T \hat{S}_2 N_3 \chi(s) \geq 0. \quad (20)$$

Construct the augmented Lyapunov-Krasovskii functional of the following form:

$$V(\delta_s, \dot{\delta}_s) = V_1(\delta_s) + V_2(\delta_s) + V_3(\delta_s, \dot{\delta}_s) \quad (21)$$

where

$$V_1(\delta_s) = \int_{s-h}^s \delta^T(s) S_1 \delta(s) ds, \ V_2(\delta_s) = \tilde{\delta}^T(t) P \tilde{\delta}(t) + \delta^T(t) Y \delta(t),$$
$$V_3(\delta_s, \dot{\delta}_s) = h\int_{-h}^0 \int_{s+\theta}^s \dot{\delta}^T(a) R_1 \dot{\delta}(a) da d\theta, \tilde{\delta}(s) = col\{\delta(s), h\sigma_1(s)\}.$$

It can be calculated that

$$\tilde{\delta}(s) = B_1\chi(s), \dot{\tilde{\delta}}(s) = B_0\chi(s). \quad (22)$$

Taking the derivative of the functional, we obtain:

$$\dot{V}_1(\delta_s) = \chi^T(s)diag\{S_1, -S_1, 0_{6n\times 6n}\}\chi(s)$$
$$= \chi^T(s)\big(\hat{S}_1 - diag\{S_1, 0_{7n\times 7n}\}\big)\chi(s)$$
$$= \chi^T(s)\hat{S}_1\chi(s) - \delta^T(s)S_1\delta(s),$$
$$\dot{V}_2(\delta_s) = 2\tilde{\delta}^T(s)P\dot{\tilde{\delta}}(s) + 2\delta^T(s)Y\dot{\delta}(s),$$
$$\dot{V}_3(\delta_s, \dot{\delta}_s) = h^2\dot{\delta}^T(s)R_1\dot{\delta}(s) - h\int_{s-h}^{s}\dot{\delta}^T(a)R_1\dot{\delta}(a)da. \quad (23)$$

By Equation (22) and Lemma 1, we have

$$\dot{V}_3(\delta_s, \dot{\delta}_s) \le \chi^T(s)\left(j_8^T h^2 R_1 j_8 - B_2^T\tilde{R}_1 B_2\right)\chi(s). \quad (24)$$

From Equation (20) together with (23)-(24), can get

$$\dot{V}(\delta_s, \dot{\delta}_s) \le \chi^T(s)\big[He\big(N_1^T P N_0 + N_4^T\hat{U}N_5 + j_1^T Y j_8\big) + \check{S}_1 + j_8^T h^2 R_1 j_8 - N_2^T\tilde{R}_1 N_2 + N_3^T\hat{S}_2 N_3\big]\chi(s)$$
$$\le \chi^T(s)\Xi_1\chi(s) + \varpi^T(s)\varpi(s). \quad (25)$$

Next, we prove that system (12)-(13) has $L_2 - L_\infty$ performance γ.

When $\varpi(t) \neq 0$, define

$$H \triangleq V(\delta_s, \dot{\delta}_s) - \int_0^s w^T(a)w(a)\,da. \quad (26)$$

When the initial state is zero, for any $\varpi(t) \in L_2[0, \infty)$, we have:

$$H = V(\delta_s, \dot{\delta}_s) - V(0) - \int_0^s \varpi^T(a)\varpi(a)\,da$$
$$= \int_0^s \big(\dot{V}(\delta_a, \dot{\delta}_a) - \varpi^T(a)\varpi(a)\big)da$$
$$= \int_0^s \chi^T(s)\Xi_1\chi(s)ds. \quad (27)$$

According to equation (24), we can known $\Xi_1 < 0$, then it follows that $H < 0$ in (25), i.e.,

$$V(\delta_s, \dot{\delta}_s) \le \int_0^s \varpi^T(a)\varpi(a)\,da. \quad (28)$$

Applying the lemma 3 to equation (23) yields the following:

$$Y - \frac{1}{\gamma^2}I^T I > 0. \quad (29)$$

From equations (26) and (28), we have

$$\delta^T(s)\delta(s) = \delta^T(s)I^T I\delta(s) < \gamma^2\delta^T(s)Y\delta(s)$$
$$\le \gamma^2 V(\delta_s, \dot{\delta}_s)$$
$$\le \gamma^2\int_0^s \varpi^T(a)\varpi(a)\,da$$
$$\le \gamma^2\int_0^\infty \varpi^T(a)\varpi(a)\,da. \quad (30)$$

Adopting the maximum value of $\parallel \delta(t) \parallel_\infty^2$, it follows that condition (14) is satisfied. Therefore, it is proved that the filtering error system (12)-(13) possesses $L_2 - L_\infty$ performance under zero initial conditions. In summary, it proves completion.

Theorem 1 provides the following linear matrix inequality (LMI) design basis for the quantized $L_2 - L_\infty$ filter:

Theorem 2 Given parameters $\gamma > 0, \mu_1 > 0, \mu_2 > 0$ and $\rho > 0$, if scalar $\varepsilon > 0$ exists, matrices $P = (P_{ij})_{3\times 3} \in S_+^{3n}$, Y, $S_1, R_1 \in S_+^n$, $M \in \mathbb{R}^{n\times m}$, and diagonal matrices $A_1, A_2 \in S_+^n$, then the following LMI can be satisfied:

$$\begin{bmatrix} Y & I \\ * & \gamma^2 I \end{bmatrix} > 0, \quad (31)$$

$$\begin{bmatrix} \Xi_2 & \rho E_1 \\ * & -\varepsilon I \end{bmatrix} < 0, \quad (32)$$

where

$$\Xi_2 = He\big(N_1^T P N_0 + j_1^T Y j_8\big) + \bar{\Sigma}_1 + \bar{\Sigma}_2 + \check{S}_1 + j_8^T h^2 R_1 j_8 - N_2^T\tilde{R}_1 N_2 + N_3^T\hat{S}_2 N_3 - j_7^T j_7 + \varepsilon E_2 E_2^T,$$

$$\bar{\Sigma}_1 = He\big[j_8^T(-\mu_1 M Q_1 + \mu_1 P_{11}W)j_1 - \mu_1 j_8^T M Q_2 j_2 + \mu_1 j_8^T P_{11} Z_1 j_5 + \mu_1 j_8^T P_{11} Z_2 j_6 + j_8^T(-\mu_1 M T_2 + \mu_1 P_{11} T_1)j_7 - \mu_1 j_8^T P_{11} j_8\big],$$

$$\bar{\Sigma}_2 = He\big[j_1^T(-\mu_2 M Q_1 + \mu_2 P_{11}W)j_1 - \mu_2 j_1^T M Q_2 j_2 + \mu_2 j_1^T P_{11} Z_1 j_5 + \mu_2 j_1^T P_{11} Z_2 j_6 + j_1^T(-\mu_2 M T_2 + \mu_2 P_{11} T_1)j_7 - \mu_2 j_1^T P_{11} j_8\big],$$

$$E_1 = [-\mu_2 M^T \quad 0_{6n\times 6n} \quad -\mu_1 M^T]^T,$$
$$E_2 = [Q_1 \quad Q_2 \quad 0_{4n\times 4n} \quad T_2 \quad 0]^T.$$

All other symbols are consistent with those in Theorem 1. Thus, with the gain $L = P_{11}^{-1}M$, the system (8)-(9) constitutes the $L_2 - L_\infty$ filter required by equations (5) and (6).

Proof: We can dedfine $U_1 = -\mu_1 P_{11}, U_2 = -\mu_2 P_{11}$. It makes the following equation hold

$$B_4^T\hat{U}B_5 = \bar{\Sigma}_1 + \bar{\Sigma}_2 - \iota\mu_1 j_8^T M Q_1 j_1 - \iota\mu_1 j_8^T M Q_2 j_2 - \iota\mu_1 j_8^T M T_2 j_7 - \iota\mu_2 j_1^T M Q_1 j_1 - \iota\mu_2 j_1^T M Q_2 j_2 - \iota\mu_2 j_1^T M T_2 j. \quad (33)$$

We can rewrite equation (16) as:

$$\hat{\Xi}_1 + He(\iota E_1 E_2^T) < 0, \quad (34)$$

where

$$\hat{\Xi}_1 = He\big(N_1^T P N_0 + j_1^T Y j_8\big) + \bar{\Sigma}_1 + \bar{\Sigma}_2 + \hat{S}_1 + j_8^T h^2 R_1 j_8 - N_2^T\tilde{R}_1 N_2 + N_3^T\hat{S}_2 N_3 - j_7^T j_7. \quad (35)$$

From Lemma 2, it can be obtained that

$$He(\iota E_1 E_2^T) \le \frac{\rho^2}{\varepsilon}E_1 E_1^T + \varepsilon E_2 E_2^T, \quad (36)$$

From this, it can be inferred that if condition

$$\hat{\Xi}_1 + \frac{\rho^2}{\varepsilon}E_1 E_1^T + \varepsilon E_2 E_2^T < 0, \quad (37)$$

Holds, then equation (34) is satisfied. Furthermore, by setting $L = P_{11}^{-1}M$ and utilizing the Schur complement property, equation (37) can be transformed into equation (32). This indicates that the conditions given in this theorem contain the conditions of Theorem 1, thereby ensuring the $L_2 - L_\infty$ performance of the filtering error system (12) - (13).

Without considering quantization, the filter equations (8)-(9) degenerate into

$$\dot{\breve{o}}(s) = W\breve{o}(s) + Z_1\xi(\breve{o}(s)) + Z_2\xi(\breve{o}(s-h)) + L[\beta(s) - \breve{\beta}(s)], \quad (38)$$

$$\breve{\beta}(s) = Q_1\breve{o}(s) + Q_2\breve{o}(s-h), \quad (39)$$

The following results can be have:

Remark 2: The LMI solver in MATLAB software can be directly used to solve the quantized $L_2 - L_\infty$ filter design method proposed in Theorem 2. First, by solving inequalities (31)–(32), we can obtain P_{11}^{-1} and M for the constant delay case, and then the filter gain matrix can be computed as $L = P_{11}^{-1}M$.

Remark 3: The LMI solver is based on interior-point algorithms, which are more efficient than traditional convex optimization algorithms. It is worth mentioning that research on LMI optimization problems remains a very hot topic to date, and it is expected that through continuous exploration and investigation, the computational speed of the algorithms can be further enhanced.

Corollary 1: For given parameters $\gamma > 0, \mu_1 > 0, \mu_2 > 0$ and $\rho > 0$, if there are matrices $P = (P_{ij})_{3\times 3} \in S_+^{3n}, \mathrm{Y}, S_1, R_1 \in S_+^n, M \in \mathbb{R}^{n\times m}$, and diagonal matrices $A_1, A_2 \in S_+^n$, then the following LMIs holds

$$\begin{bmatrix} Y & I \\ * & \gamma^2 I \end{bmatrix} > 0, \quad (40)$$

$$\Xi_3 < 0, \quad (41)$$

where

$$\Xi_3 = He(N_1^T P N_0 + j_1^T Y j_8) + \bar{\Sigma}_1 + \bar{\Sigma}_2 + \breve{S}_1 + j_8^T h^2 R_1 j_8 - N_2^T \tilde{R}_1 N_2 + N_3^T \hat{S}_2 N_3 - j_7^T j_7,$$

and all other symbols are consistent with those in Theorem 1 and Theorem 2. So, considering the neural network (5)-(6) and set the gain of filter system (38)-(39) to $L = P_{11}^{-1}M$. Under the zero initial conditions and when $\iota = 0$, the corresponding filtering error systems (12) - (13) satisfy the given $L_2 - L_\infty$ performance metric γ.

4. Numerical Simulation

In this part, we provide a numerical example with simulation to verify that the proposed quantized $L_2 - L_\infty$ filter design method can be effectively applied to the studied constant time-delay neural network system.

Based on equation (5)-(6), we select a two-neuron neural network modeled with constant time delay and the following parameters [11]

$$W = \begin{bmatrix} -1 & 0 \\ 0 & -3.5 \end{bmatrix}, Z_1 = \begin{bmatrix} -1 & 0.4 \\ 0 & -0.1 \end{bmatrix}, Z_2 = \begin{bmatrix} 0.2 & -0.8 \\ 0.4 & 0.5 \end{bmatrix},$$

$$Q_1 = [1 \quad 0], Q_2 = [0.5 \quad 1], \mathrm{T}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mathrm{T}_2 = 1, \varpi(t) = e^{-t}.$$

Select the activation function as shown in $\xi(o(s)) = [\tanh(o_1(s)) \quad \tanh(o_2(s))]^T$. And $\xi(o(s))$ satisfies equation (7) with $L_\xi = 1$. Furthermore, the initial conditions are set as

$$o(s) = \begin{bmatrix} -3.2 \\ 1.5 \end{bmatrix}, \breve{o}(s) = \begin{bmatrix} 2.4 \\ -2.1 \end{bmatrix}, s \in [-1, 0].$$

Assuming $\rho = 0.2$, setting $h = 1, \mu_1 = 0.3$ and $\mu_2 = 0.8, \gamma = 0.26$ is calculated, and we can solve the LMIs in equations (31)-(32) to obtain

$$P_{11} = \begin{bmatrix} 14.1821 & 3.0971 \\ 3.0971 & 23.4384 \end{bmatrix}, M = \begin{bmatrix} 11.5654 \\ -18.1833 \end{bmatrix}.$$

Therefore, according to Theorem 2, the system (8)-(9) with gain matrix

$$L = P_{11}^{-1}M = \begin{bmatrix} 1.0142 \\ -0.9098 \end{bmatrix}$$

is the desired quantized $L_2 - L_\infty$ filter.

Table 2. When $\rho = 0.2$, the Interference Attenuation Level γ Changes with Time Delay h

h	0.5	1	1.5
γ	0.197	0.259	0.568

Table 3. When $h = 1$, the Interference Attenuation Level γ Changes with Quantization Density ρ

ρ	0.1	0.2	0.3
γ	0.123	0.259	0.414

From Tables 2 and 3, it can be seen that variations in both the time delay h and the quantization density ρ have a notable impact on the minimum $L_2 - L_\infty$ disturbance attenuation level γ. Specifically, Table 2 shows that when the quantization density is fixed at $\rho = 0.2$ an increase in the time delay h from 0.5 to 1.5 leads to a significant rise in γ from 0.197 to 0.568 , indicating that a larger time delay degrades the disturbance attenuation performance. Similarly, Table 3 reveals that for a fixed time delay $h = 1$, a higher quantization density ρ (i.e., finer quantization) results in a larger γ, meaning that the filtering performance becomes worse as the quantization becomes denser. In both cases, the disturbance attenuation level γ increases monotonically with h and ρ, implying that the system's ability to suppress external disturbances weakens as the delay grows or the quantization becomes finer. Moreover, the relationship is nonlinear: the increase in γ is more pronounced at larger values of h and ρ. These observations highlight the trade-off between network-induced imperfections (delay and quantization) and filtering performance, providing practical guidance for system design. The simulation results of the $L_2 - L_\infty$ filter design are shown in Figures 1–4. Specifically, Figures 1 and 2 illustrate the time trajectories of the system states $o_1(s)$ and $o_2(s)$ along with their respective estimates $\breve{o}_1(s)$ and $\breve{o}_2(s)$. It can be observed that each estimated state closely follows its true counterpart, demonstrating the satisfactory tracking performance of the designed filter. The trajectory of the filtering error $\delta(s)$ is shown in Figure 3, which remains bounded and exhibits decaying behavior, indicating that the error system is stable. With zero initial conditions, the variation curve of $L(s) = \sqrt{\{sup_{0\le t}\delta^T(s)\delta(s)\}\Big/\Big\{\int_0^K \varpi^T(s)\varpi(s)ds\Big\}}$ is depicted in Figure 4. From Figure 4, the maximum value of $L(s)$ over the simulation horizon is found to be $L(s)_{max} = 0.0569$, which is significantly smaller than the predefined $L_2 - L_\infty$ performance $\gamma = 0.26$. This confirms that the designed quantized $L_2 - L_\infty$ filter achieves the required disturbance attenuation level, thereby verifying its effectiveness.

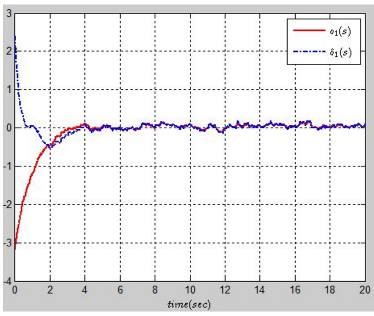

Figure 1. Trajectory of State $o_1(s)$ and Its Estimate $\breve{o}_1(s)$

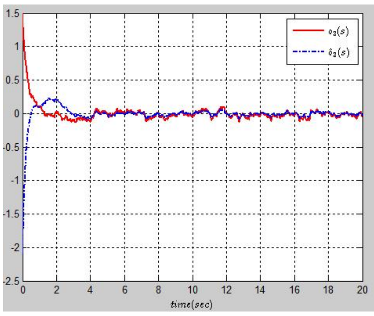

Figure 2. Trajectory of State $o_2(s)$ and Its Estimate $\breve{o}_2(s)$

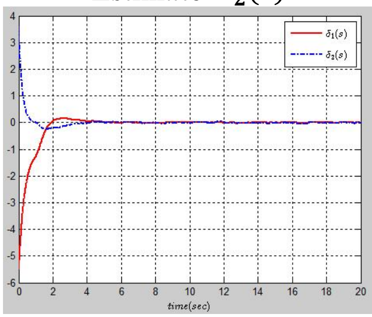

Figure 3. Trajectory of State Filtering Error $\delta(s)$

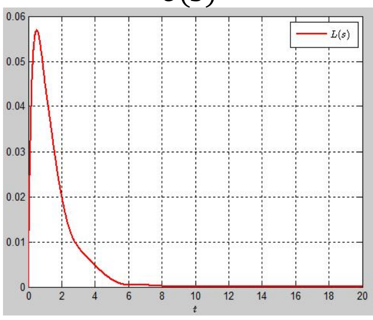

Figure 4. The Trajectory of $L(s)$

5. Conclusion

This paper investigates the $L_2 - L_\infty$ filtering problem for constant time-delay neural networks with quantized output. First, to fully exploit the characteristics of the constant delay and the quantization effect, we construct an augmented Lyapunov functional that incorporates additional system state information. By jointly employing the Bessel–Legendre inequality (which provides a tighter bound than the traditional Jensen-based inequality) and the free weighted matrix method by bring in flexible relaxation variables, a sufficient criterion is established to assure that the error system can attain the prescribed $L_2 - L_\infty$ performance. This criterion effectively reduces conservatism while maintaining numerical tractability. Based on this criterion, an explicit linear matrix inequality (LMI) based design method is further developed for the desired quantized $L_2 - L_\infty$ filter. The acquisition of the filter gain matrix can be directly attributed to solving a set of linear matrix inequalities, such as using the standard solver in MATLAB. Finally, the applicability and reduced conservatism of the filter design method proposed in the article are demonstrated through a numerical example. The results also confirmed the effectiveness of our method for the studied neural network systems with output quantization and constant time delay.

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